

New results on maximal partial line spreads in $\text{PG}(5, q)$

Maurizio Iurlo

Abstract

In this work, we prove the existence of maximal partial line spreads in $\text{PG}(5, q)$ of size $q^3 + q^2 + kq + 1$, with $1 \leq k \leq \left\lfloor \frac{q^3 - q^2}{q+1} \right\rfloor$, k an integer. Moreover, by a computer search, we do this for larger values of k , for $q \leq 7$. Again by a computer search, we find the sizes for the largest maximal partial line spreads and many new results for $q \leq 5$.

Keywords: Maximal partial line spreads - Computer search

AMS Classification: 51E14

1 Introduction

A *partial line spread* \mathcal{F} in $\text{PG}(N, q)$, the projective space of dimensions N over the Galois field $\text{GF}(q)$ of order q , is a set of pairwise skew lines. We say that \mathcal{F} is *maximal* if it cannot be extended to a larger partial line spread. A *line spread* in $\text{PG}(N, q)$, N odd, is a set of pairwise skew lines covering the space.

Here, we consider a hyperplane \mathcal{H} of $\text{PG}(5, q)$ and a partial line spread \mathcal{F} of \mathcal{H} , of size $q^3 + 1$, which is the largest size of MPS in $\text{PG}(4, q)$. So, in \mathcal{H} there are exactly q^2 *holes*, that is points not over the lines of \mathcal{F} .

We start by adding to \mathcal{F} a set of q^2 pairwise skew lines, not of \mathcal{H} , covering the holes of \mathcal{H} , obtaining a maximal partial spread in $\text{PG}(5, q)$ of size $q^3 + q^2 + 1$. We proceed by depriving the obtained maximal partial spread in $\text{PG}(5, q)$ of some lines of \mathcal{H} and adding $q+1$ pairwise skew lines not of \mathcal{H} for each removed line. We do this through a theoretical way for

every value of q , and by a computer search for $q \leq 7$. More precisely we prove that for every q there are maximal partial line spreads of the sizes $q^3 + q^2 + kq + 1$, for any $1 \leq k \leq \left\lfloor \frac{q^3 - q^2}{q+1} \right\rfloor$, k an integer, while by a computer search we get cardinalities $q^3 + q^2 + kq + 1$, for larger values of k , for $q \leq 7$.

Moreover, by a computer search, we find the sizes for the largest maximal partial spreads in $\text{PG}(5, q)$ for $q \leq 5$: we prove that the largest maximal partial line spreads have deficiency $\delta = 2$ for $q = 2$, $\delta = \frac{q+3}{2}$ for $q = 3, 5$ and $\delta = \sqrt{q} + 1$ for $q = 4$.

Again by a computer search, we find many new results for $q \leq 5$.

2 Known results

Maximal partial line spreads (from now on MPS) in $\text{PG}(5, q)$ have been investigated by several authors.

In [1], Beutelspacher proved the existence of MPS of size $q^3 + q^2 + 1$. In [6], Govaerts showed that this size is the smallest size for MPS in $\text{PG}(5, q)$.

We recall that a spread in $\text{PG}(N, q)$, N odd, have size

$$\frac{q^{N+1} - 1}{q^2 - 1}.$$

Also, we have the following results.

Theorem 2.1. ([5]) *For any q odd and for q even, $q > q_0$, in $\text{PG}(N, q)$, $N \geq 5$ odd, there exist maximal partial spreads of any size between $9Nq^{N-2} \log q$ and $\frac{q^{N+1}-1}{q^2-1} - q + 1$.*

We remark that the above interval is empty for $q \leq 243$. So, for $q \leq 7$, the only known sizes are $q^3 + q^2 + 1$ and the size of a spread.

Upper bounds for the size of the largest example of MPS were achieved by Govaerts and Storme [7, 8].

If \mathcal{S} is a partial line spread of $\text{PG}(N, q)$, N odd, and $|\mathcal{S}| = \frac{q^{N+1}-1}{q^2-1} - \delta$, then we say that \mathcal{S} has deficiency δ .

Theorem 2.2. ([8]) *Let $\epsilon = 2$ in the case $q = 2$ and let $q + \epsilon$ be the size of the smallest non-trivial blocking sets in $\text{PG}(2, q)$ in the case $q > 2$. Suppose N odd and $\delta < \epsilon$. Unless $\delta = 0$ there exist no maximal partial line spreads with deficiency δ in $\text{PG}(N, q)$.*

Corollary 2.3. ([8]) *Let \mathcal{S} be a partial s -spread of $\text{PG}(N, q)$, N odd, of deficiency $\delta > 0$. Then*

1. ([4]) $\delta \geq \sqrt{q} + 1$ when q is square;
2. ([3]) $\delta \geq c_p q^{2/3} + 1$, when $q = p^h$, h odd, $h > 2$, p prime, $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$;
3. ([2]) $\delta \geq (q + 3)/2$ when q is an odd prime.

Theorem 2.4. ([7]) *Let \mathcal{S} be a maximal partial line spread of $\text{PG}(N, q)$, N odd, $q > 16$, of deficiency $0 < \delta < q^{5/8}/\sqrt{2} + 1$. Then $\delta \equiv 0 \pmod{\sqrt{q}+1}$ and the set of holes of \mathcal{S} is the disjoint union of subgeometries $\text{PG}(2s+1, \sqrt{q})$. Moreover, $\delta \geq 2(\sqrt{q} + 1)$ when $q > 4$.*

3 A geometric construction of maximal partial line spreads in $\text{PG}(5, q)$

We start by proving the following lemma, similarly to Lemma 2.1 in [10].

Lemma 3.1. *In $\text{PG}(5, q)$, q a prime power, let S be a hyperplane and X a point of S . Let \mathcal{L} be a set of lines not of S , not through X , not two of them meeting outside S , and such that $|\mathcal{L}| < q^3$. Then there is a line through X not of S and skew to every line of \mathcal{L} .*

Let \mathcal{H} be a hyperplane of $\text{PG}(5, q)$ and let \mathcal{F} be a largest partial line spread of \mathcal{H} , the size of which is $q^3 + 1$. So, in \mathcal{H} there are exactly q^2 holes, that is points not over the lines of \mathcal{F} . By Lemma 3.1 it follows that there exists a set of q^3 mutually disjoint lines of $\text{PG}(5, q)$, not of \mathcal{H} , covering any set of q^3 points of \mathcal{H} . So, we choose a line set \mathcal{F}' of q^2 lines, not in \mathcal{H} , covering the q^2 holes of \mathcal{H} . In this way we get a maximal partial spread $\mathcal{F} \cup \mathcal{F}'$ of $\text{PG}(5, q)$, with size $q^3 + q^2 + 1$. At this point we deprive \mathcal{F} of a line r_1 and cover the points of r_1 through $q + 1$ lines, $r_1^1, r_1^2, \dots, r_1^{q+1}$, not of \mathcal{H} , whose existence is guaranteed by above lemma, obtaining the MPS

$$\mathcal{F}_1 = ((\mathcal{F} \cup \mathcal{F}') - r_1) \cup \{r_1^1, r_1^2, \dots, r_1^{q+1}\}.$$

We proceed depriving \mathcal{F}_1 of a line r_2 of \mathcal{F} and, by using the above lemma again, we find $q + 1$ mutually skew lines $r_2^1, r_2^2, \dots, r_2^{q+1}$, not of \mathcal{H} and covering r_2 . The line set

$$\mathcal{F}_2 = (\mathcal{F}_1 - r_2) \cup \{r_2^1, r_2^2, \dots, r_2^{q+1}\}$$

is a MPS of size $q^3 + q^2 + 2q + 1$.

We proceed up to deprive \mathcal{F} of $n = \left\lfloor \frac{q^3 - q^2}{q+1} \right\rfloor$ lines, obtaining a MPS

$$\mathcal{F}_n = (\mathcal{F}_{n-1} - r_n) \cup \{r_n^1, r_n^2, \dots, r_n^{q+1}\},$$

with $q^3 + q^2 + nq + 1$ lines.

So, we give a class of maximal partial line spreads of sizes $q^3 + q^2 + kq + 1$, with $1 \leq k \leq \left\lfloor \frac{q^3 - q^2}{q+1} \right\rfloor$.

So we get the following theorem.

Theorem 3.2. *In $\text{PG}(5, q)$, q a prime power, there are maximal partial line spreads of size $q^3 + q^2 + kq + 1$, for every integer $k = 1, 2, \dots, \left\lfloor \frac{q^3 - q^2}{q+1} \right\rfloor$.*

We note that, for $N = 5$ and $q \geq 2$,

$$9Nq^{N-2} \log q > q^3 + q^2 + \left\lfloor \frac{q^3 - q^2}{q+1} \right\rfloor q + 1,$$

and so the theorem gives $\left\lfloor \frac{q^3 - q^2}{q+1} \right\rfloor$ unknown cardinalities for any q .

4 Computer search of maximal partial line spreads in $\text{PG}(5, q)$

We firstly report, in the Table 1, for the investigated values of q , the minimum sizes $q^3 + q^2 + 1$, the maximum sizes (see Theorem 2.2 and Corollary 2.3), the size $\frac{q^6 - 1}{q^2 - 1} - q + 1$ (see Theorem 2.1) and the sizes of the spreads $\frac{q^6 - 1}{q^2 - 1}$.

In the Table 2, we report the obtained cardinalities of type $q^3 + q^2 + k \cdot q + 1$ of MPS, for $q \leq 7$, by the Theorem 3.2 and by the first algorithm for the computer search.

In the Table 2, $k_{\min} = 1$ and $|\mathcal{F}_{\min}| = q^3 + q^2 + q + 1$; k_{\max} is the maximum value of k and $|\mathcal{F}_{\max}| = q^3 + q^2 + k_{\max} \cdot q + 1$.

So we get the following theorem:

Theorem 4.1. *In $\text{PG}(5, q)$, $q \leq 7$, there are maximal partial line spreads of size $q^3 + q^2 + kq + 1$, for every integer $k = 1, 2, \dots, \left\lfloor \frac{4}{9}q^3 \right\rfloor$.*

Table 1

q	$q^3 + q^2 + 1$	δ	Max size	$\frac{q^6-1}{q^2-1} - q + 1$	Spread
2	13	≥ 2	≤ 19	20	21
3	37	≥ 3	≤ 88	89	91
4	81	≥ 3	≤ 270	270	273
5	151	≥ 4	≤ 647	647	651
7	393	≥ 5	≤ 2446	2445	2451

Table 2

q	Theorem 3.2		Computer search
	$ \mathcal{F}_{min} $	$k_{max} \rightarrow \mathcal{F}_{max} $	$k_{max} \rightarrow \mathcal{F}_{max} $
2	15	$1 \rightarrow 15$	$3 \rightarrow 19$
3	40	$4 \rightarrow 49$	$12 \rightarrow 73$
4	85	$9 \rightarrow 117$	$28 \rightarrow 193$
5	156	$16 \rightarrow 231$	$59 \rightarrow 446$
7	400	$36 \rightarrow 645$	$163 \rightarrow 1534$

We now report all the obtained cardinalities by computer search, that we denote by $b + k_m^n \cdot q$, where $b = q^3 + q^2 + 1$ and $m \leq k \leq n$, k integer. The underlined values are those not known and that cannot be obtained through the Theorem 3.2.

- $q = 2$: 13, 15, 16 – 19, 21.

- $q = 3$: $37 + k_0^4 \cdot 3$, $37 + k_5^7 \cdot 3$, 60 – 88, 91.

- $q = 4$: $81 + k_0^9 \cdot 4$, $81 + k_{10}^{19} \cdot 4$, 159 – 270, 273.

- $q = 5$: $151 + k_0^{16} \cdot 5$, $151 + k_{17}^{49} \cdot 5$, 399 – 647, 651.

- $q = 7$: $393 + k_0^{36} \cdot 7$, $393 + k_{37}^{163} \cdot 7$.

We note that all smallest MPS have sizes of type $q^3 + q^2 + kq + 1$, while, starting from a certain size, there are all the possible sizes.

By the previous result, we get the following theorem (see also Table 1).

Theorem 4.2. *In $\text{PG}(5, q)$, $q \leq 5$, the largest maximal partial line spreads have deficiency $\delta = 2$ for $q = 2$, $\delta = \frac{q+3}{2}$ for $q = 3, 5$ and $\delta = \sqrt{q} + 1$ for $q = 4$.*

The Theorem 2.1, for $N = 5$ and $q = 3, 4, 5$, can be modified as follows.

Theorem 4.3. *In $\text{PG}(5, q)$, $q = 3, 4, 5$, there exist maximal partial line spreads of any size between $2q^3 \log q$ and $\frac{q^6-1}{q^2-1} - \delta$.*

References

- [1] A. Beutelspacher, Blocking sets and partial spreads in finite projective spaces, *Geom. Dedicata* 9(4) (1980), 425–449.
- [2] A. Blokhuis, On the size of a blocking set in $\text{PG}(2, p)$, *Combinatorica* 14(1) (1994), 111–114.
- [3] A. Blokhuis, L. Storme and T. Szőnyi, Lacunary polynomials, multiple blocking sets and Baer subplanes, *J. London Math. Soc.* (2) 60(2) (1999), 321–332.
- [4] A. A. Bruen, Baer subplanes and blocking sets, *Bull. Amer. Math. Soc.* 76 (1970), 342–344.
- [5] A. Gács and T. Szőnyi, On maximal partial spreads in $\text{PG}(n, q)$, *Des. Codes Cryptogr.* 29(1) (2003), 123–129.
- [6] P. Govaerts, Small maximal partial t -spreads, *Bull. Belg. Math. Soc. Simon Stevin* 12(4) (2005), 607–615.
- [7] P. Govaerts and L. Storme, On a particular class of minihypers and its applications. II. Improvements for q square, *J. Combin. Theory Ser. A* 97(2) (2002), 369–393.
- [8] P. Govaerts and L. Storme, On a particular class of minihypers and its applications. I. The result for general q , *Des. Codes Cryptogr.* 28(1) (2003), 51–63.

- [9] M. Iurlo and S. Rajola, *A new method to construct maximal partial spreads of smallest sizes in $\text{PG}(3, q)$* . In: Error-correcting codes, finite geometries and cryptography, Contemp. Math., vol. 523, Amer. Math. Soc., Providence, RI, 2010, 89–107.
- [10] S. Rajola and M. Iurlo, A new class of maximal partial line spreads in $\text{PG}(4, q)$, *J. Combin. Math. Combin. Comput.* 95 (2015), 193–199.
- [11] M. Iurlo and S. Rajola, A computer search of maximal partial spreads in $\text{PG}(3, q)$, to appear in *Ars Combinatoria*, *arXiv:1011.5338*.

Maurizio Iurlo
Largo dell’Olgiata, 15
00123 Roma
Italy
maurizio.iurlo@istruzione.it
<http://www.maurizioiurlo.com>